ON EXPLICIT SOLUTIONS OF INTERVAL LINEAR PROGRAMS

BY

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ABSTRACT

An interval linear program is the problem of maximizing $\{(c, x): a \leqq A x \leqq b\}$ for given matrix A and vectors a, b and c. The explicit (noniterative) solutions of interval programs given here, extend earlier results of Ben-Israel and Charnes.

Introduction. An interval linear program, abbreviated IP, is defined as a linear program of the form:

(1) maximize (c,x) s.t. $a \leqq Ax \leqq b$ where the vectors a, b, c and the matrix A are given.

This problem was introduced in [2] and solved explicitly in the feasible bounded case with A of full row rank. The general case was studied in [4], [5] and solved iteratively.

The results of [2] are extended in this paper to IP's with matrices of arbitrary rank.

Preliminaries and notations. The following notations are used:

R^n *the n-dimensional real vector space*

R^{mxn} *the space of mxn real matrices*

R_r^{mxn} $= \{X \in R^{mxn}: \text{rank } X = r\}$

For any $x, y \in R^n$:

$x \geqq y$ denotes $x_i \geqq y_i \quad (i = 1, \cdots, n)$

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$$(x, y) = \sum_{i=1}^{n} x_i y_i$$

For any subspace $L \subset R^n$:

P_L the *perpendicular projection* on L

$x + L$ the *manifold* $\{x + l : l \in L\}$

For any $A \in R^{mxn}$:

A^t the *transpose* of A

$R(A)$ the *range space* of A

$N(A)$ the *null space* of A

$A\{1\} = \{T \in R^{nxm}: ATA = A\}$

A^+ the *generalized inverse* of A, [3].

e the vector of ones, with dimension clear from context.

The $IP(1)$ is denoted by

$$IP(a, b, c, A) \tag{2}$$

$IP(a, b, c, A)$ is *feasible* if

$F = \{x \in R^n: a \leqq Ax \leqq b\} \neq \emptyset$ and *bounded* if

$\max\{(c, x): x \in F\} < \infty$. A feasible $IP(a, b, c, A)$ is bounded if, and only if

$$c \in R(A^t) \quad [2]. \tag{3}$$

The mapping $\eta: R^m x R^m x R^m \to R^m$ is defined by

$$\eta(u, v, w) = (\eta_i) \qquad (i = 1, \cdots, m)$$

where

$$\eta_i = \begin{cases} u_i & \text{if } w_i < 0 \\ v_i & \text{if } w_i > 0 \\ \theta_i u_i + (1 - \theta_i) v_i & \text{if } w_i = 0 \end{cases} \tag{4}$$

and $0 \leqq \theta_i \leqq 1$.

For any $w \in R^m, \eta(u, v, w)$ is linear in $\begin{pmatrix} u \\ v \end{pmatrix}$.

In what follows we occasionally refer to $x + L$, $\eta(u, v, w)$ in the singular sense as "vector", "solution", etc., although in general these are sets of vectors.

Results. Conditions for an explicit solution of $IP(a, b, c, A)$ are given in:

THEOREM 1:

Assumptions: $a, b \in R^m$; $c \in R^n$; $A \in R^{m \times n}$ *are given such that* $IP(a,b,c,A)$ *is feasible and bounded.*

$T \in A\{1\}$ *is arbitrary.*

$$x_0 = TP_{R(A)}\eta(a,b,(TP_{R(A)})^t c) + N(A) \tag{5}$$

Conclusion: x_0 *is an optimal solution of* $IP(a,b,c,A)$ *if and only if* x_0 *is a feasible solution of* $IP(a,b,c,A)$.

PROOF.

Only if: Obvious.

If: Substituting

$$u = Ax, \ x = TP_{R(A)}u + N(A) \tag{6}$$

and using (3) it follows that (2) is equivalent to the problem:

$$\max((TP_{R(A)})^t c, \ u) \tag{7}$$

$$\text{s.t.}$$

$$a \leqq u \leqq b \tag{8}$$

$$u \in R(A). \tag{9}$$

The optimality of x_0 follows from (6) and the fact that

$$u_0 = P_{R(A)}\eta_0, \ \eta_0 = \eta(a,b,(TP_{R(A)})^t c) \tag{10}$$

is an optimal solution of (7) (8) (9), which will now be proved.

From

$$ATP_{R(A)} = ATAA^+ = AA^+ = P_{R(A)}$$

and x_0 a feasible solution of (2) it follows that u_0 satisfies (8). Therefore u_0 is a feasible solution of (7) (8) (9). To prove that it is optimal we show that its value $((TP_{R(A)})^t c, u_0)$ equals the maximum value of the less restricted problem (7) (8): The optimal solution of (7) (8) is clearly η_0 and its maximal value is:

$$\begin{aligned} ((TP_{R(A)})^t c, \eta_0) &= (P_{R(A)}T^t c, \eta_0) \\ &= (P_{R(A)}T^t c, P_{R(A)}\eta_0) \\ &= ((TP_{R(A)})^t c, u_0). \end{aligned}$$ □

x_0 defined by (5) is independent of the particular $T \in A\{1\}$ used in its definition. In particular (5) can be rewritten for $T = A^+$ as

(5′) $$x_0 = A^+\eta(a,b,A^{+t}c) + N(A).$$

This follows from:

$$\begin{aligned}(TP_{R(A)})^t c &= P^t_{R(A)}T^t c\\ &= A^{+t}A^tT^tc\\ &= A^{+t}A^tT^tA^td,\end{aligned}$$

for some d since $c \in R(A^t)$ by (3)

$$\begin{aligned}&= A^{+t}A^td \text{ since } T \in A\{1\}\\ &= A^{+t}c\end{aligned}$$

and from the fact that

$$A\{1\} = A^+ + \{W : AWA = 0\}$$

which implies that:

$$\begin{aligned}x_0 &= TP_{R(A)}\eta_0 + N(A)\\ &= (A^+ + W)P_{R(A)}\eta_0 + N(A) \text{ where } AWA = 0\\ &= A^+\eta_0 + WAA^+\eta_0 + N(A)\\ &= A^+\eta_0 + N(A) \text{ since } WAA^+\eta_0 \in N(A).\end{aligned}$$

For A of full row rank i.e.

$$AA^+ = P_{R(A)} = I$$

x_0 defined by (5) is always feasible since

$$\begin{aligned}Ax_0 &= AA^+\eta(a,b,A^{+t}c), \quad \text{by (5′)}\\ &= \eta(a,b,A^{+t}c)\end{aligned}$$

which, by definition (4), lies in the interval $[a,b]$.

This special case is the main result of [2]:

COROLLARY 1. *Let $IP(a,b,c,A)$ be feasible and bounded, $A \in R_m^{m\times n}$ and $T \in A\{1\}$. Then the optimal solution of $IP(a,b,c,A)$ is*

(11) $$x_0 = T\eta(a,b,T^tc) + N(A)$$

□

Another class of IP's which can always be solved explicity is considered in:

COROLLARY 2.

Assumptions: $g, h \in R^m$; $c \in R^n$; $A \in R^{m\times n}$ *are given such that*

(12) $$IP(g - P_{N(A^t)}\eta, h - P_{N(A^t)}\eta, c, A)$$

is feasible and bounded where $T \in A\{1\}$ *is arbitrary and*

$$\eta = \eta(g, h, (TP_{R(A)})^t c)$$

Conclusion: *The optimal solution of* (12) *is*:

(13) $$x = TP_{R(A)}\eta + N(A).$$

PROOF. By Theorem 1 it suffices to show that (13) is a feasible solution of (12), i.e. that

(14) $$g - P_{N(A^t)}\eta \leqq Ax \leqq h - P_{N(A^t)}\eta$$

but

$$Ax = ATP_{R(A)}\eta = P_{R(A)}\eta$$

so (14) becomes

$$g - P_{N(A^t)}\eta \leqq P_{R(A)}\eta \leqq h - P_{N(A^t)}\eta$$

or finally

$$g \leqq \eta \leqq h$$

□

Since the explicit solution of

(12) $$IP(g - P_{N(A^t)}\eta,\ h - P_{N(A^t)}\eta,\ c, A)$$

is available, we may approximate general $IP(a, b, c, A)$ by problems of class (12). For the choice $g = a$ and $h = b$ Corollary 2 gives:

COROLLARY 3.

Assumptions: $a, b \in R^m$; $c \in R^n$; $A \in R^{m \times n}$ *are given such that*

(15) $$IP(a - P_{N(A^t)}\eta_0,\ b - P_{N(A^t)}\eta_0,\ c,\ A)$$

is feasible and bounded where $T \in A\{1\}$ *is arbitrary and*

$$\eta_0 = \eta(a, b, (TP_{R(A)})^t c).$$

Conclusion: *The optimal solution of* (15) *is*

$$x_0 = TP_{R(A)}\eta_0 + N(A)$$ □

The above results are now applied to a standard linear program:

$$\text{(16)} \qquad \begin{aligned} &\text{maximize } (c,x) \\ &\text{s.t. } Ax = b, \\ &\qquad x \geqq 0 \end{aligned}$$

where $A \in R^{mxn}$; $b \in R^m$; $c \in R^n$ are given. An additional constraint:

$$\text{(17)} \qquad x \leqq u,$$

where $u \in R^n$ is a vector whose components u_j are positive and sufficiently large, is now adjoined to (16). If (16) is bounded, then (17) is redundant for sufficiently large u_j, $j = 1,\cdots,n$. Conversely, if for all $u_j > 0$, $j = 1,\cdots,n$ the optimal solution of (16) (17) is a function of some u_j then (16) is unbounded. Now $Ax = b$ is equivalent to

$$\text{(18)} \qquad x = Tb + Ny$$

where $T \in A\{1\}$, N is any matrix whose columns span $N(A)$ i.e. $R(N) = N(A)$, y determined by x, T, N. Substituting (18) in (16)(17) we get, by ignoring the constant term (c, Tb) in the functional, the problem:

$$\begin{aligned} &\text{maximize } (c, Ny) \\ &\text{s.t.} \\ &-Tb \leqq Ny \leqq u - Tb \end{aligned}$$

i.e.

$$\text{(19)} \qquad IP(-Tb,\ u - Tb,\ N^t c,\ N).$$

From Theorem 1 we conclude that for any $S \in N\{1\}$

$$\text{(20)} \qquad y = SP_{N(A)}\eta(-Tb,\ u - Tb,\ (SP_{N(A)})^t N^t c) + N(N)$$

is an optimal solution of (19) if and only if it is a feasible solution.

Now we note that

$$\begin{aligned} NSP_{N(A)} &= NSNN^+ && \text{since } R(N) = N(A) \\ &= NN^+ && \text{since } S \in N\{1\} \\ &= P_{N(A)}. \end{aligned}$$

Therefore (20) substituted in (18) gives

$$\text{(21)} \qquad \begin{aligned} x &= Tb + P_{N(A)}\eta(-Tb,\ u - Tb,\ P_{N(A)}c) \\ &= Tb + P_{N(A)}[-Tb + \eta(0, u, P_{N(A)}c)], \end{aligned}$$

by the linearity of η,

$$= P_{R(A^t)}Tb + P_{N(A)}\eta(0, u, P_{N(A)}c)$$

$$= A^+ b + P_{N(A)}\eta\ (0, u, P_{N(A)}c),$$

since $P_{R(A^t)}Tb = A^+ATb = A^+b$.

Collecting these results we get:

THEOREM 2.

Assumptions: $A \in R^{m\times n}$, $b \in R^m$, $u > 0$ in R^n *are such that the problem*

$$\text{maximize } (c, x) \quad \text{s.t.} \quad Ax = b, \quad 0 \leqq x \leqq u \tag{22}$$

is feasible.

$$x^* = A^+b + P_{N(A)}\eta(0, u, P_{N(A)}c) \tag{21}$$

Conclusion: x^* *is an optimal solution of* (22) *if and only if it is a feasible solution.*

□

Examples

EXAMPLE 1:

$$\text{Maximize } x_1 - x_2$$
$$\text{s.t.} \quad 0 \leqq x_1 + x_2 \leqq 2$$
$$1 \leqq x_1 \quad \leqq \frac{5}{2}$$
$$-1 \leqq \quad x_2 \leqq 3$$

Here

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and as in [2] we compute

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_{R(A)} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

therefore

$$TP_{R(A)} = \frac{1}{3}\begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$(TP_{R(A)})^t c = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\eta_0 = \eta(a, b, (TP_{R(A)})^t c) = \eta\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ \frac{5}{2} \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2 - 2\theta \\ \frac{5}{2} \\ -1 \end{bmatrix}, \ 0 \leqq \theta \leqq 1$$

$$u_0 = P_{R(A)}\eta_0 = \frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 - 2\theta \\ \frac{5}{2} \\ -1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} \frac{11}{2} - 4\theta \\ 8 - 2\theta \\ -2\theta - \frac{5}{2} \end{bmatrix}, \quad 0 \leqq \theta \leqq 1$$

u_0 satisfies (8) i.e.

$$a = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \leqq \frac{1}{3}\begin{bmatrix} \frac{11}{2} - 4\theta \\ 8 - 2\theta \\ -2\theta - \frac{5}{2} \end{bmatrix} \leqq \begin{bmatrix} 2 \\ \frac{5}{2} \\ 3 \end{bmatrix} = b$$

if and only if $\theta = \frac{1}{4}$, in which case $u_0 = \begin{pmatrix} \frac{3}{2} \\ \frac{5}{2} \\ -1 \end{pmatrix}$

Since $N(A) = \{0\}$, the optimal solution is by (5)

$$x_0 = TP_{R(A)}\eta_0 = Tu_0$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{bmatrix} \frac{3}{2} \\ \frac{5}{2} \\ -1 \end{bmatrix} = \begin{pmatrix} \frac{5}{2} \\ -1 \end{pmatrix}$$

EXAMPLE 2: Maximize x_1

s.t. $x_1 + x_2 = \alpha \quad (\alpha > 0)$

$x_1, x_2 \geqq 0$

Here $A = (1 \quad 1)$, $A^+ = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$P_{N(A)} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \ P_{N(A)}c = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

By (21) we compute

$$\begin{aligned} x &= A^+b + P_{N(A)}\eta(0, u, P_{N(A)}c) \\ &= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\alpha + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\eta\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) \\ &= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\alpha + \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \alpha + u_1 \\ \alpha - u_1 \end{pmatrix} \end{aligned}$$

which is feasible for $u_1 = \alpha$. Therefore the optimal solution is $x = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$.

EXAMPLE 3: Maximize x_1

$$\text{s.t.} \qquad x_1 - x_2 = \alpha \quad (\alpha > 0)$$

$$x_1, x_2 \geqq 0$$

This problem is unbounded, but Theorem 2 can still be used to describe the optimal ray:

Here

$$A = (1, -1), \quad A^+ = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$P_{N(A)} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad P_{N(A)}c = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and by (21)

$$\begin{aligned} x &= A^+b + P_{N(A)}\eta(0, u, P_{N(A)}c) \\ &= \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\alpha + \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\eta\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \\ &= \frac{1}{2}\begin{pmatrix} \alpha \\ -\alpha \end{pmatrix} + \begin{pmatrix} \lambda \\ \lambda \end{pmatrix}, \quad \lambda = \frac{u_1 + u_2}{2} \end{aligned}$$

which is feasible for $\lambda \geqq \alpha/2$.

Since the set $\left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}: x_1 - x_2 = \alpha,\ x_1 \geqq 0,\ x_2 \geqq 0\right\}$ is unbound, and the bounds u_1, u_2 cannot be finite. The optimal solution is therefore:

$$\lim_{\mu\to\infty}\begin{pmatrix}\mu+\alpha\\ \mu\end{pmatrix}$$

EXAMPLE 4: Maximize $-2x_1 + x_2$

s.t. $x_1 - x_2 = 1$

$x_1, x_2 \geqq 0$

Here

$$A = (1, -1),\ A^+ = \frac{1}{2}\begin{pmatrix}1\\ -1\end{pmatrix}$$

$$P_{N(A)}c = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}-2\\ 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1\\ -1\end{pmatrix}$$

$$\eta(0, u, P_{N(A)}c) = 0$$

Therefore by (21)

$$x^* = A^+b = \frac{1}{2}\begin{pmatrix}1\\ -1\end{pmatrix},\ \text{unfeasible.}$$

This problem cannot therefore be solved explicitly in this form, as is the case whenever $P_{N(A)}c \leqq 0$, $A^+b \ngeqq b$.

EXAMPLE 5: Maximize $x_1 + x_2$

s.t. $x_1 + x_2 + x_3 = 1$

$x_1, x_2, x_3 \geqq 0$

There are infinitely many optimal solutions, i.e. all points of the form

$$x = \begin{pmatrix}\theta\\ 1-\theta\\ 0\end{pmatrix}, \quad 0 \leqq \theta \leqq 1.$$

It is interesting to see how we get this by using (21):
Writing $y_1 = x_1 + x_2$, $y_2 = x_3$ the problem is seen to be equivalent to:

Maximize y_1

s.t. $y_1 + y_2 = 1$

$y_1, y_2 \geqq 0$

whose solution by example 2 is:

$$\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$$

Therefore the optimal solution is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \theta \\ 1-\theta \\ 0 \end{pmatrix}, \quad 0 \leqq \theta \leqq 1.$$

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